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SEIDEL ENERGY OF GENERALIZED COMPLEMENTS OF GRAPHS

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Abstract. Let $P = \{V_1, V_2, \cdots, V_k\}$ be a partition of vertex set V of a graph G. The $k-$ complement of G denoted by G_k^P is defined as follows: for all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add edges between V_i and V_j which are not in G. The graph G is k-self complementary with respect to P if $G_k^P \cong G$. The k(i)-complement $G_{k(i)}^P$ of a graph G with respect to P is defined as follows: for all $V_r \in P$, remove edges inside V_r and add edges which are not in V_r. Any graph G is k(i)-self complementary if $G_{k(i)}^P \cong G$. In this paper, we study Seidel energy of generalized complements of some families of graph. An effort is made to throw some light on showing variation in Seidel energy due to changes in a partition of the graph.

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1. Introduction

All graphs considered in this paper will be assumed to be simple, finite, and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of a graph G, denoted by $\overline{G}$ has the same vertex set as that of G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The graph G is said to be a self-complementary graph if G is isomorphic to its complement $\overline{G}$ [6]. Over the years other graph complements like generalized graph complements have been defined and studied. The concept of generalized complements was introduced by E. Sampathkumar et.al [10] as follows.

Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of vertex set V of G. The $k-$ complement of G denoted by G_k^P is defined as follows: for all $V_i, V_j \in P$ and $i \neq j$, remove the edges between V_i and V_j and add edges between V_i and V_j which are not in G. The graph G is k-self complementary with respect to P if $G_k^P \cong G$. The k(i)-complement $G_{k(i)}^P$ of a graph G with respect to P is defined as follows: for all $V_r \in P$, remove edges inside V_r and add edges which are not in V_r. Any graph G is k(i)-self complementary if $G_{k(i)}^P \cong G$. For more details on generalized complements of graphs we study [2, 8, 9, 10].

The adjacency matrix of a graph G is denoted by $A(G) = [a_{ij}]$, is a real symmetric matrix of order $n \times n$, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise, for all v_i, $v_j \in \{v_1, v_2, \ldots, v_n\}$ of G. The energy of the graph was first defined by Ivan Gutman in 1978 as the sum of absolute eigenvalues of graph G [4].

The Seidel matrix denoted by $S(G) = [s_{ij}]$ is a real symmetric matrix of order n, where

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$s_{ij} = 1$ if the vertices v_i and v_j are non adjacent, $s_{ij} = -1$ if the vertices v_i and v_j are adjacent and $s_{ij} = 0$ otherwise.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $S(G)$. The characteristic polynomial of $S(G)$ is denoted by $\phi(S(G), \lambda) = det(\lambda I - S(G))$. The Seidel energy of a graph is defined as

$$SE(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The motivation to define the energy of graphs arose from quantum chemistry in 1930. E. Hückel presented chemical applications of graph theory in his molecular orbital theory where eigenvalues of graphs take place. In quantum chemistry, nonsaturated hydrocarbon is represented by a graph. The energy levels of electrons in such a molecule are eigenvalues of a graph. The carbon atoms and chemical bonds between them in a hydrocarbon system denote vertices and edges, respectively, in a molecular graph. A lot of work has been done on graph theory, chemical graph theory, and graph energies. For more information about graph energy we studied [1, 3, 5, 7]. In the present article, we computed the Seidel energy of generalized complements of a graph.

2. Seidel energy of generalized complements of some graphs

In this section, we obtain Seidel energy of generalized complements of some classes of graphs such as cycle, complete graph, star graph, crown graph, complete bipartite graph, and cocktail party graph.

The following results are proved by showing $AZ = \lambda Z$ for certain vectors Z and by making use of the fact that geometric and algebraic multiplicity of each characteristic value λ is the same, as $S((G)_k^P)$ is real and symmetric.

Theorem 2.1. *The Seidel energy of 2-complement of a complete graph is* $2(n-1)$.

Proof. Let $P = \{V_1, V_2\}$ be a partition of vertex set of K_n, where $|V_1| = m$ and $|V_2| = n - m$. The Seidel matrix of $(K_n)_2^P$ is

$$S((K_n)_2^P) = \left[\begin{array}{c|c} (I-J)_{m\times m} & J_{m\times n-m} \\ \hline J_{n-m\times m} & (I-J)_{n-m\times n-m} \end{array}\right]_{n\times n}.$$

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$, be a characteristic vector which is partitioned conformally with $S((K_n)_2^P)$.
Now,

$$\left(\lambda I - S((K_n)_2^P)\right)\begin{bmatrix} X_m \\ Y_{n-m} \end{bmatrix} = \begin{bmatrix} ((\lambda-1)I + J)\,X - JY \\ -JX + ((\lambda-1)I + J)\,Y \end{bmatrix}. \tag{1}$$

Case 1: Let $Y = \mathbf{1}_{n-m}$ and $X = \dfrac{(n-m)(\lambda-1)\mathbf{1}_m}{\lambda^2 + \lambda(m-2) - (m-1)}$, where λ be any root of characteristic polynomial of $S((K_n)_2^P)$.

From equation (1),

$$\frac{(\lambda+m-1)(n-m)(\lambda-1)\mathbf{1}_m}{\lambda^2+\lambda(m-2)-(m-1)} - J_{(m\times n-m)}\mathbf{1}_{(n-m)} = 0, and$$

$$- J_{(n-m\times m)}\frac{(n-m)(\lambda-1)\mathbf{1}_m}{\lambda^2+\lambda(m-2)-(m-1)} + [(\lambda-1)I+J]_{(n-m\times n-m)}\mathbf{1}_{n-m}$$
$$= \frac{(\lambda+n-m-1)(\lambda+m-1)-m(n-m)}{(\lambda+m-1)} = 0.$$

Therefore 1 and $-(n-1)$ are eigenvalues of $S((K_n)_2^P)$ each with multiplicity one.
Case 2: Let $X = X_i$ be an eigenvector having first element 1 and i^{th} element -1, for $i = 2, 3, \dots, m$ and rest of the entries be zero, and $Y = 0_{n-m}$.
From equation (1), $-JX_i + ((\lambda-1)I+J)\,0_{n-m} = 0$ and $((\lambda-1)I+J)\,X_i - J0_{n-m} = (\lambda-1)X_i = 0$. Thus 1 is characteristic value with multiplicity $(m-1)$.
Case 3: Let $Y = Y_j$ be an eigenvector having first element 1 and j^{th} element -1, for $j = 2, 3, \dots, n-m$ and rest of the entries be zero, and $X = 0_m$.
From equation (1), $((\lambda-1)I+J)\,0_m - JY_j = 0$ and $-J0_m + ((\lambda-1)I+J)\,Y_j = (\lambda-1)Y_j = 0$. Thus 1 is characteristic value with multiplicity $(n-m-1)$.
Thus Seidel spectrum of $(K_n)_2^P$ is

$$Spec((K_n)_2^P) = \begin{pmatrix} -(n-1) & 1 \\ 1 & (n-1) \end{pmatrix}.$$

Hence, $SE((K_n)_2^P) = 2(n-1)$. □

Theorem 2.2. *The Seidel energy of 2-complement of a star graph is* $2(n-1)$.

Proof. Let $P = \{V_1, V_2\}$ be a partition of S_n. Then Seidel matrix of $(S_n)_2^P$ is

$$S((S_n)_2^P) = \left[\begin{array}{c|c} (J-I)_{n-m+1\times n-m+1} & -J_{n-m+1\times m-1} \\ \hline -J_{m-1\times n-m+1} & (J-I)_{m-1\times m-1} \end{array}\right]_{n\times n}.$$

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be a characteristic vector which is partitioned conformally with $S((S_n)_2^P)$.
Now,

$$(2) \qquad \left(\lambda I - S((S_n)_2^P)\right)\begin{bmatrix} X_m \\ Y_{n-m} \end{bmatrix} = \begin{bmatrix} ((\lambda+1)I-J)\,X + JY \\ JX + ((\lambda+1)I-J)\,Y \end{bmatrix}.$$

Case 1: Let $Y = \mathbf{1}_{m-1}$ and $X = \dfrac{-(m-1)(\lambda+1)\mathbf{1}_{n-m+1}}{\lambda^2-\lambda(n-m-1)-(n-m)}$, where λ be any root of characteristic polynomial of $S((S_n)_2^P)$. From equation (2), $\dfrac{(m-1)(\lambda-n+m)(\lambda+1)\mathbf{1}_{n-m+1}}{\lambda^2-\lambda(n-m-1)-(n-m)} + J_{(n-m+1\times m-1)}\mathbf{1}_{(m-1)} = 0$, and

$$J_{(m-1\times n-m+1)}\frac{-(m-1)(\lambda+1)\mathbf{1}_{n-m+1}}{\lambda^2-\lambda(n-m-1)-(n-m)}+[(\lambda+1)I-J]_{(m-1\times m-1)}\mathbf{1}_{m-1}=$$
$$\frac{(\lambda-n+m)(\lambda-m+2)-(n-m+1)(m-1)}{(\lambda-n+m)}=0.$$

Therefore -1 and $(n-1)$ are eigenvalues of $S((S_n)_k^P)$ each with multiplicity one.
Case 2: Let $X=X_i$ be an eigenvector having first element 1 and i^{th} element -1, for $i=2,3,\dots,n-m+1$ and rest of the entries be zero, and $Y=0_{m-1}$.
From equation (2), $JX_i+((\lambda+1)I-J)\,0_{m-1}=0$ and $((\lambda+1)I-J)\,X_i+J0_{n-m}$ is $(\lambda+1)X_i=0$. Thus -1 is the characteristic value with multiplicity $(n-m)$.
Case 3: Let $Y=Y_j$ be an eigenvector having first element 1 and j^{th} element -1, for $j=2,3,\dots,m-1$ and rest of the entries be zero, and $X=0_{n-m+1}$. From equation (2), $((\lambda+1)I-J)\,0_{n-m+1}+JY_j=0$, and $J0_{n-m+1}+((\lambda+1)I-J)\,Y_j=(\lambda+1)Y_j=0$. Thus -1 is the characteristic value with multiplicity $(m-2)$.
The Seidel spectrum of $(S_n)_2^P$ is

$$Spec((S_n)_2^P)=\begin{pmatrix}(n-1) & -1\\ 1 & (n-1)\end{pmatrix}.$$

Hence, $SE((S_n)_2^P)=2(n-1)$. □

Remark $(S_n)_2^P$ and $(K_n)_2^P$ are non-cospectral Seidel equienergetic graphs.

Theorem 2.3. *For crown graph S_{2n} with a partition $P=\{V_1,V_2,\dots,V_n\}$ and $\langle V_i\rangle=\overline{K_2}$ for all $i=1,2,\dots,n$, the Seidel energy of $(S_{2n})_n^P$ is $2(2n-1)$.*

Proof. Let $P=\{V_1,V_2,\dots,V_n\}$ be a partition of S_{2n} such that $\langle V_i\rangle=\overline{K_2}$. The Seidel matrix of $(S_{2n})_n^P$ is

$$S((S_{2n})_n^P)=\left[\begin{array}{c|c}(I-J)_{n\times n} & J_{n\times n}\\ \hline J_{n\times n} & (I-J)_{n\times n}\end{array}\right]_{2n\times 2n}.$$

Let $Z=\begin{bmatrix}X\\Y\end{bmatrix}$ be a characteristic vector which is partitioned conformally with $S((S_{2n})_n^P)$.
Now,

$$(3)\qquad \left(\lambda I-S((S_{2n})_n^P)\right)\begin{bmatrix}X_n\\Y_n\end{bmatrix}=\begin{bmatrix}((\lambda-1)I+J)\,X-JY\\-JX+((\lambda-1)I+J)\,Y\end{bmatrix}.$$

Case 1: Let $X=\mathbf{1}_n$ and $Y=\dfrac{n(\lambda-1)\mathbf{1}_n}{\lambda^2-\lambda(n-2)-(n-1)}$, where λ be any root of characteristic polynomial of $S((S_{2n})_n^P)$. From equation (3), $-J_{(n\times n)}\mathbf{1}_{(n)}+\dfrac{n(\lambda-1)(\lambda+n-1)\mathbf{1}_n}{\lambda^2+\lambda(n-2)-(n-1)}=0$, and $((\lambda-1)I+J)_{(n\times n)}\mathbf{1}_n-\dfrac{-n(\lambda-1)J_{n\times n}\mathbf{1}_n}{\lambda^2+\lambda(n-2)-(n-1)}=\dfrac{(\lambda+n-1)(\lambda+n-1)-n^2}{(\lambda+n-1)}=0$.
Therefore 1 and $-(2n-1)$ are eigenvalues of $S((S_{2n})_k^P)$ each with multiplicity one.
Case 2: Let $X=X_i$ be an eigenvector having first element 1 and i^{th} element -1, for

$i = 2, 3, \ldots, n$ and rest of the entries be zero, and $Y = 0_n$.
From equation (3), $-JX_i + ((\lambda-1)I + J)\,0_n = 0$, and $((\lambda-1)I + J)\,X_i - J0_n = (\lambda-1)X_i = 0$. Thus 1 is characteristic value with multiplicity $(n-1)$.
Case 3: Let $Y = Y_j$ be an eigenvector having first element 1 and j^{th} element -1, for $j = 2, 3, \ldots, n$ and rest of the entries be zero, and $X = 0_n$.
From equation (3), $((\lambda-1)I + J)\,0_n - JY_j = 0$ and $-J0_n + ((\lambda-1)I + J)\,Y_j = (\lambda-1)Y_j = 0$. Thus 1 is characteristic value with multiplicity $(n-1)$.
The Seidel spectrum of $(S_{2n})_n^P$ is

$$Spec((S_{2n})_n^P) = \begin{pmatrix} -(2n-1) & 1 \\ 1 & (2n-1) \end{pmatrix}.$$

Hence, $SE((S_{2n})_n^P) = 2(2n-1)$. □

Theorem 2.4. *Let $P = \{V_1, V_2\}$ be a partition of crown graph S_{2n} of vertex set $V = \{v_1, v_2, \ldots, v_n, u_{n+1}, u_{n+2}, \ldots, u_{2n}\}$ such that $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_{n+1}, u_{n+2}, \ldots, u_{2n}\}$ then Seidel energy of $(S_{2n})_2^P$ is $6(n-1)$.*

Proof. Let $P = \{V_1, V_2\}$ be a partition of S_{2n} such that $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_{n+1}, u_{n+2}, \ldots, u_{2n}\}$. The Seidel matrix of $(S_{2n})_2^P$ is

$$S((S_{2n})_2^P) = \left[\begin{array}{c|c} (J-I)_{n\times n} & (J-2I)_{n\times n} \\ \hline (J-2I)_{n\times n} & (J-I)_{n\times n} \end{array}\right]_{2n\times 2n}.$$

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be a characteristic vector is partitioned conformally with $S((S_{2n})_2^P)$. Now,

$$(4) \qquad \left(\lambda I - S((S_{2n})_2^P)\right) \begin{bmatrix} X_n \\ Y_n \end{bmatrix} = \begin{bmatrix} ((\lambda+1)I - J)\,X + (-J + 2I)Y \\ (-J + 2I)X + ((\lambda+1)I - J)\,Y \end{bmatrix}.$$

Case 1: Let $X = \mathbf{1}_n$ and $Y = \dfrac{(n-2)(\lambda+1)\mathbf{1}_n}{\lambda^2 - \lambda(n-2) - (n-1)}$, where λ be any root of characteristic polynomial of $S((S_{2n})_2^P)$.
From equation (4), $(-J+2I)_{(n\times n)}\mathbf{1}_{(n)} + \dfrac{(n-2)(\lambda-n+1)(\lambda+1)\mathbf{1}_n}{\lambda^2 - \lambda(n-2) - (n-1)} = 0$ and $[(\lambda+1)I - J]_{(n\times n)}\mathbf{1}_n + (-J+2I)_{(n\times n)}\dfrac{(n-2)(\lambda+1)\mathbf{1}_n}{\lambda^2 - \lambda(n-2) - (n-1)} = \dfrac{(\lambda-n+1)(\lambda-n+1) - (n-2)^2}{(\lambda-n+1)} = 0$.
Therefore, 1 and $(2n-3)$ are eigenvalues of $S((S_{2n})_2^P)$ each with multiplicity one.
Case 2: Let $Y = Y_j$ be an eigenvector having first element 1 and j^{th} element -1, for $j = 2, 3, \ldots, n$ and rest of the entries be zero, and $X = \dfrac{-2(\lambda-n+1)Y_j}{\lambda^2 - \lambda(n-2) - (n-1)}$.
From equation (4), $((\lambda+1)I - J)\,\dfrac{-2(\lambda-n+1)Y_j}{\lambda^2 - \lambda(n-2) - (n-1)} + (-J + 2I)Y_j = 0$, and $(-J + 2I)\dfrac{-2(\lambda-n+1)Y_j}{\lambda^2 - \lambda(n-2) - (n-1)} + ((\lambda+1)I - J)\,Y_j$ is $((\lambda+1)^2 - 4)Y_j = 0$. Thus 1 and -3 are characteristic values each with multiplicity $(n-1)$.

The Seidel spectrum of $(S_{2n})_2^P$ is

$$Spec((S_{2n})_2^P) = \begin{pmatrix} (2n-3) & -3 & 1 \\ 1 & (n-1) & n \end{pmatrix}.$$

Hence, $SE((S_{2n})_2^P) = 6(n-1)$. □

Theorem 2.5. *Let $P = \{V_1, V_2, V_3\}$ be a partition of $K_{m,n}$ of vertex set $V = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \ldots, u_n\}$ such that $V_1 = \{v_1, v_2, \ldots, v_m\}$, $V_2 = \{u_1, u_2, \ldots, u_{\frac{n}{2}}\}$ and $V_3 = \{u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \ldots, u_n\}$ then $SE((S_{2n})_3^P) = \dfrac{8m+8n-12}{3}$.*

Proof. The Seidel matrix of $(K_{m,n})_3^P$ is

$$S((K_{m,n})_3^P) = \left[\begin{array}{c|c|c} (J-I)_{\frac{m+n}{3}\times\frac{m+n}{3}} & J_{\frac{m+n}{3}\times\frac{m+n}{3}} & J_{\frac{m+n}{3}\times\frac{m+n}{3}} \\ \hline J_{\frac{m+n}{3}\times\frac{m+n}{3}} & (J-I)_{\frac{m+n}{3}\times\frac{m+n}{3}} & -J_{\frac{m+n}{3}\times\frac{m+n}{3}} \\ \hline J_{\frac{m+n}{3}\times\frac{m+n}{3}} & -J_{\frac{m+n}{3}\times\frac{m+n}{3}} & (J-I)_{\frac{m+n}{3}\times\frac{m+n}{3}} \end{array}\right]_{m+n\times m+n}.$$

Let $Z = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ be a characteristic vector which is partitioned conformally with $S((K_{m,n})_k^P)$. Now,

$$(5) \qquad \left(\lambda I - S((K_{m,n})_3^P)\right)\begin{bmatrix} X_{\frac{m+n}{3}} \\ Y_{\frac{m+n}{3}} \\ Z_{\frac{m+n}{3}} \end{bmatrix} = \begin{bmatrix} ((\lambda+1)I - J)\,X - JY - JZ \\ -JX + ((\lambda+1)I - J)\,Y + JZ \\ -JX + JY + ((\lambda+1)I - J)\,Z \end{bmatrix}.$$

Case 1: Let $X = \mathbf{1}_{\frac{m+n}{3}}$, $Y = 0_{\frac{m+n}{3}}$ and $Z = \dfrac{\left(\frac{m+n}{3}\right)\mathbf{1}_{\frac{m+n}{3}}}{\lambda - \left(\frac{m+n}{3}\right)+1}$ where λ be any root of characteristic polynomial of $S((K_{m,n})_3^P)$.

From equation (5), $-J\mathbf{1}_{\frac{m+n}{3}} + J0_{\frac{m+n}{3}} + \dfrac{\left(\frac{m+n}{3}\right)\mathbf{1}_{\frac{m+n}{3}}}{\lambda - \left(\frac{m+n}{3}\right)+1} = 0$, and

$-J\mathbf{1}_{\frac{m+n}{3}} + J0_{\frac{m+n}{3}} + \dfrac{J\left(\frac{m+n}{3}\right)\mathbf{1}_{\frac{m+n}{3}}}{\lambda - \left(\frac{m+n}{3}\right)+1} = \lambda - 2\left(\dfrac{m+n}{3}\right) + 1 = 0.$

Therefore $\dfrac{2m+2n-3}{3}$ is an eigenvalue of $S((K_{m,n})_3^P)$ with multiplicity one.

Case 2: Let $X = X_i$, $Y = 0_{\frac{m+n}{3}}$ and $Z = 0_{\frac{m+n}{3}}$. From equation (5), $-JX_i + ((\lambda+1)I - J)\,0_{\frac{m+n}{3}} + J0_{\frac{m+n}{3}} = 0$ and $((\lambda+1)I - J)\,X_i - J0_{\frac{m+n}{3}} - J0_{\frac{m+n}{3}} = (\lambda+1)X_i = 0$. Thus -1 is characteristic value with multiplicity $\left(\dfrac{m+n}{3} - 1\right)$.

Case 3: Let $Y = Y_i$, $X = 0_{\frac{m+n}{3}}$ and $Z = 0_{\frac{m+n}{3}}$. From equation (5), $-J0_{\frac{m+n}{3}} + JY_i + ((\lambda+1)I - J)\,0_{\frac{m+n}{3}} = 0$ and $-J0_{\frac{m+n}{3}} + ((\lambda+1)I - J)\,Y_i + J0_{\frac{m+n}{3}} = (\lambda+1)Y_i = 0$. Thus -1 is characteristic value with multiplicity $\left(\dfrac{m+n}{3} - 1\right)$.

Case 4: Let $Z = Z_i$, $X = 0_{\frac{m+n}{3}}$ and $Y = 0_{\frac{m+n}{3}}$. From equation (5), $-J0_{\frac{m+n}{3}} +$

$((\lambda+1)I-J)\,0_{\frac{m+n}{3}}+JZ_i=0$ and $-J0_{\frac{m+n}{3}}+J0_{\frac{m+n}{3}}+((\lambda+1)I-J)$ $Z_i=(\lambda+1)Z_i=0$. Thus -1 is characteristic value with multiplicity $\left(\frac{m+n}{3}-1\right)$.

Case 5: Let $Y=\mathbf{1}_{\frac{m+n}{3}}$, $Z=\mathbf{1}_{\frac{m+n}{3}}$ and $X=\dfrac{2\left(\frac{m+n}{3}\right)(\lambda+1)\mathbf{1}_{\frac{m+n}{3}}}{\lambda^2-\lambda\left(\frac{m+n-6}{3}\right)-\left(\frac{m+n-3}{3}\right)}$. From equation (5), $\dfrac{2((\lambda+1)I-J)\left(\frac{m+n}{3}\right)(\lambda+1)\mathbf{1}_{\frac{m+n}{3}}}{\lambda^2-\lambda\left(\frac{m+n-6}{3}\right)-\left(\frac{m+n-3}{3}\right)}-J\mathbf{1}_{\frac{m+n}{3}}+-J\mathbf{1}_{\frac{m+n}{3}}=0$ and $\dfrac{-2J\left(\frac{m+n}{3}\right)(\lambda+1)\mathbf{1}_{\frac{m+n}{3}}}{\lambda^2-\lambda\left(\frac{m+n-6}{3}\right)-\left(\frac{m+n-3}{3}\right)}+J\mathbf{1}_{\frac{m+n}{3}}+((\lambda+1)I-J)\mathbf{1}_{\frac{m+n}{3}}=\lambda^2-\lambda\left(\frac{m+n-6}{3}\right)-2\left(\frac{m+n}{3}\right)^2-\left(\frac{m+n}{3}\right)+1=0$. Thus $-\left(\dfrac{m+n+3}{3}\right)$ and $\dfrac{2m+2n-3}{3}$ are the characteristic values each with multiplicity 1.
The Seidel spectrum of $(K_{m,n})_3^P$ is

$$Spec((K_{m,n})_3^P)=\begin{pmatrix} -\left(\dfrac{m+n+3}{3}\right) & \dfrac{2m+2n-3}{3} & -1\\ 1 & 2 & m+n-3\end{pmatrix}.$$

Hence, $SE((K_{m,n})_3^P)=\dfrac{8m+8n-12}{3}$. □

Theorem 2.6. *For a cocktail party graph $K_{n\times 2}$ with $V=\{v_1,v_2,\dots,v_n,u_1,u_2,\dots,u_n\}$ and $P=\{V_1,V_2\}$, where $V_1=\{v_1,v_2,\dots,v_n\}$ and $V_2=\{u_1,u_2,\dots,u_n\}$, the Seidel energy is $6(n-1)$.*

Proof. Let $K_{n\times 2}$ be a cocktail party graph of order $2n$, and $(K_{n\times 2})_2^P$ be k-complement of $K_{n\times 2}$ with respect to the partition $P=\{V_1,V_2\}$. The Seidel matrix of $(K_{n\times 2})_2^P$ is

$$S((K_{n\times 2})_2^P)=\left[\begin{array}{c|c}(I-J)_{n\times n} & (J-2I)_{n\times n}\\ \hline (J-2I)_{n\times n} & (I-J)_{n\times n}\end{array}\right]_{2n\times 2n}.$$

Let $Z=\begin{bmatrix}X\\Y\end{bmatrix}$ be a characteristic vector which is partitioned conformally with $S((K_{n\times 2})_2^P)$.
Now,

$$(6)\qquad \left(\lambda I-S((K_{n\times 2})_2^P)\right)\begin{bmatrix}X_n\\Y_n\end{bmatrix}=\begin{bmatrix}((\lambda-1)I+J)\,X+(-J+2I)Y\\(-J+2I)X+((\lambda-1)I+J)\,Y\end{bmatrix}.$$

Case 1: Let $X=X_i$ and $Y=\dfrac{-2(\lambda+n-1)X_i}{\lambda^2+\lambda(n-2)-(n-1)}$, where λ be any root of characteristic polynomial of $S((K_{n\times 2})_2^P)$.
From equation (6), $(2I-J)X_i+\dfrac{((\lambda-1)I+J)(-2(\lambda+n-1)X_i)}{\lambda^2+\lambda(n-2)-(n-1)}=0$ and $((\lambda-1)I+J)X_i+\dfrac{(2I-J)(-2(\lambda+n-1)X_i)}{\lambda^2+\lambda(n-2)-(n-1)}=\dfrac{((\lambda-1)^2-4)X_i}{(\lambda-1)}=0$.
Therefore -1 and 3 are eigenvalues each with multiplicity $(n-1)$.
Case 2: Let $Y=\mathbf{1}_n$ and $X=\dfrac{(n-2)(\lambda-1)\mathbf{1}_n}{\lambda^2+\lambda(n-2)-(n-1)}$ where λ be any root of characteristic polynomial of $S((K_{n\times 2})_2^P)$.

From equation (6), $\dfrac{(\lambda+n-1)(n-2)(\lambda-1)\mathbf{1}_n}{\lambda^2+\lambda(n-2)-(n-1)}+(2-n)\mathbf{1}_n=0$ and
$\dfrac{(2-n)(n-2)(\lambda-1)\mathbf{1}_n}{\lambda^2+\lambda(n-2)-(n-1)}+((\lambda-+n-1)\mathbf{1}_n=\dfrac{(\lambda+n-1)^2-(n-2)^2}{(\lambda+n-1)}=0$. Thus -1 and $-(2n-3)$ are eigenvalues each with multiplicity 1.
The Seidel spectrum of $(K_{n\times2})_2^P$ is

$$Spec((K_{n\times2})_2^P)=\begin{pmatrix}-(2n-3) & 3 & -1\\ 1 & n-1 & n\end{pmatrix}.$$

Hence, $SE((K_{n\times2})_2^P)=6(n-1)$.

□

Theorem 2.7. *For a cocktail party graph $K_{n\times2}$ with $P=\{V_1,V_2,\ldots,V_n\}$, where $\langle V_i\rangle=K_2$ for all $i=1,2,\ldots,n$, the Seidel energy is $7n-10$.*

Proof. Let $P=\{V_1,V_2,\ldots,V_n\}$ be a partition with each partite $\langle V_i\rangle=K_2$ for all $i=1,2,\ldots,n$. We have

$$S((K_{n\times2})_n^P)=\begin{bmatrix}
0 & -1 & 1 & 1 & \cdots & 1 & 1 & -1 & 1 & 1 & 1 & \cdots & 1 & 1\\
-1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & -1 & 1 & 1 & \cdots & 1 & 1\\
1 & 1 & 0 & -1 & \cdots & 1 & 1 & 1 & 1 & -1 & 1 & \cdots & 1 & 1\\
1 & 1 & -1 & 0 & \cdots & 1 & 1 & 1 & 1 & 1 & -1 & \cdots & 1 & 1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & 1 & 1 & \cdots & 0 & -1 & 1 & 1 & 1 & 1 & \cdots & -1 & 1\\
1 & 1 & 1 & 1 & \cdots & -1 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & -1\\
-1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & -1 & 1 & 1 & \cdots & 1 & 1\\
1 & -1 & 1 & 1 & \cdots & 1 & 1 & -1 & 0 & 1 & 1 & \cdots & 1 & 1\\
1 & 1 & -1 & 1 & \cdots & 1 & 1 & 1 & 1 & 0 & -1 & \cdots & 1 & 1\\
1 & 1 & 1 & -1 & \cdots & 1 & 1 & 1 & 1 & -1 & 0 & \cdots & 1 & 1\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & 1 & 1 & \cdots & -1 & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & -1\\
1 & 1 & 1 & 1 & \cdots & 1 & -1 & 1 & 1 & 1 & 1 & \cdots & -1 & 0
\end{bmatrix}_{2n\times2n}$$

the above matrix is of the form

$$A=\left[\begin{array}{c|c}A_0 & A_1\\ \hline A_1 & A_0\end{array}\right]_{2n\times2n}.$$

Consider $det(\lambda I-A)$.
As matrix A is a block symmetric matrix, the eigenvalues of A are the union of eigenvalues of matrices A_0+A_1 and A_0-A_1. First we shall find eigenvalues of A_0-A_1.
Consider $(\lambda I-(A_0-A_1))$,
Step 1: Replacing R_i by R_i-R_{i+1}, for all $i=1,3,5,\ldots,(n-1)$, we obtain

$$|\lambda I-(A_0-A_1)|=\begin{vmatrix}
(\lambda-3) & -(\lambda-3) & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
2 & \lambda-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & (\lambda-3) & -(\lambda-3) & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 2 & \lambda-1 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
 & 0 & 0 & 0 & 0\cdots & (\lambda-3) & -(\lambda-3) & 0 & 0\\
 & 0 & 0 & 0 & 0\cdots & 2 & \lambda-1 & 0 & 0\\
 & 0 & 0 & 0 & 0\cdots & 0 & 0 & (\lambda-3) & -(\lambda-3)\\
 & 0 & 0 & 0 & 0\cdots & 0 & 0 & 2 & \lambda-1
\end{vmatrix}_{n\times n}$$

Step 2: By replacing C_i by $C_i - C_{i-1}$, for $i = 2, 4, 6, \ldots, n$, the determinant reduces to

$$|\lambda I - (A_0 - A_1)| = \begin{vmatrix} (\lambda-3) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 2 & \lambda+1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & (\lambda-3) & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & \lambda+1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (\lambda-3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 2 & \lambda+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (\lambda-3) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2 & \lambda+1 \end{vmatrix}_{n\times n}$$

Expanding along rows $i = 1, 3, 5, \ldots, (n-1)$ it reduces to $(\lambda-3)^{\frac{n}{2}}(\lambda+1)^{\frac{n}{2}}$.

Now to find eigenvalues of $A_0 + A_1$ we follow the below steps.

Consider $|\lambda I - (A_0 + A_1)|$,

Step 1: Replacing R_i by $R_i - R_{i+1}$, for all $i = 1, 3, 5, \ldots, (n-1)$, the determinant becomes

$$|\lambda I - (A_0 + A_1)| = \begin{vmatrix} (\lambda+1) & -(\lambda+1) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \lambda+1 & -2 & -2 & \cdots & -2 & -2 & -2 & -2 \\ 0 & 0 & (\lambda+1) & -(\lambda+1) & \cdots & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & \lambda+1 & \cdots & -2 & -2 & -2 & -2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (\lambda+1) & -(\lambda+1) & 0 & 0 \\ -2 & -2 & -2 & -2 & \cdots & 0 & \lambda+1 & -2 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (\lambda+1) & -(\lambda+1) \\ -2 & -2 & -2 & -2 & \cdots & -2 & -2 & 0 & \lambda+1 \end{vmatrix}_{n\times n}$$

Step 2: Now replacing C_i by $C_i - C_{i-1}$, for $i = 2, 4, 6, \ldots, n$. The determinant simplifies to

$$|\lambda I - (A_0 + A_1)| = \begin{vmatrix} (\lambda+1) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \lambda+1 & -2 & -4 & \cdots & -2 & -4 & -2 & -4 \\ 0 & 0 & (\lambda+1) & 0 & \cdots & 0 & 0 & 0 & 0 \\ -2 & -4 & 0 & \lambda+1 & \cdots & -2 & -4 & -2 & -4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (\lambda+1) & 0 & 0 & 0 \\ -2 & -4 & -2 & -4 & \cdots & 0 & \lambda+1 & -2 & -4 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & (\lambda+1) & 0 \\ -2 & -4 & -2 & -4 & \cdots & -2 & -4 & 0 & \lambda+1 \end{vmatrix}_{n\times n}$$

Expanding the determinant along the rows $i = 1, 3, 5, \ldots, (n-1)$ it reduces to

$$|\lambda I - (A_0 + A_1)| = (\lambda+1)^{\frac{n}{2}} \begin{vmatrix} (\lambda+5) & -(\lambda+5) & 0 & 0 & \cdots & 0 & 0 \\ 0 & (\lambda+5) & -(\lambda+5) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (\lambda+5) & -(\lambda+5) \\ -4 & -4 & -4 & -4 & \cdots & -4 & (\lambda+1) \end{vmatrix}_{\frac{n}{2}\times\frac{n}{2}}$$

Step 3: By replacing R_i by $R_i - R_{i+1}$ for all $i = 1, 2, 3, \ldots, \frac{n}{2} - 1$.

$$|\lambda I - (A_0 + A_1)| = (\lambda+1)^{\frac{n}{2}} \begin{vmatrix} (\lambda+5) & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (\lambda+5) & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (\lambda+5) & 0 \\ -4 & -8 & -12 & -16 & \cdots & -(2n-4) & (\lambda+1) \end{vmatrix}_{\frac{n}{2}\times\frac{n}{2}}$$

Step 4: Now replace C_i by $C_i + C_{i-1}$ for all $i = 1, 2, 3 \ldots, \frac{n}{2}$, it reduces to

$$|\lambda I-(A_0+A_1)| = (\lambda+1)^{\frac{n}{2}}(\lambda+5)^{\frac{n}{2}-1}\begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 0\\ -4 & -8 & -12 & -16 & \cdots & -(2n-4) & (\lambda+1) \end{vmatrix}_{\frac{n}{2}\times\frac{n}{2}}$$

On expanding along the rows R_i, $i=1,2,3,\ldots,\frac{n}{2}-1$ we get $(\lambda+1)^{\frac{n}{2}}(\lambda+5)^{\frac{n}{2}-1}(\lambda-2n+5)$. The Seidel spectrum of $(K_{n\times 2})_2^P$ is

$$Spec((K_{n\times 2})_n^P) = \begin{pmatrix} 2n-5 & -5 & 3 & -1\\ 1 & \frac{n}{2}-1 & \frac{n}{2} & n \end{pmatrix}.$$

Hence, $SE((K_{n\times 2})_n^P) = 7n-10$.

□

3. Conclusion

In this paper, Seidel energy of generalized complements of some families of graphs with respect to different partitions was obtained, and studied variation of Seidel energy due to changes in the partition of graphs. Finding the Seidel energy of other classes of graphs with respect to different partitions is an open area of research.

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